End Semester Examination 2019

IVth Semester B.Sc. Mathematics Honours

Core Course IX

Full Marks: 50 **Time: 2 hours**

Unit-I

1. Answer any one of the following questions: **$1 \times 1 = 1$**

a) Find $d(S)$.

$$S = \{(x,y,z): |x| \leq 2, |y| \leq 1, |z-2| \leq 2\}$$

where $d(S)$ denotes the diameter of S.

b) Show that , if $f(x,y)$ is differentiable at (x,y), then it must be continuous there.

2. Answer any three of the following questions: **$3 \times 4 = 12$**

a) Use Lagrange's method of undetermined multiplier to determine the point (or points) on the ellipse,$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so that the tangent line to the point (or points) form with the co-ordinate axes a triangle of smallest area.

b) Find the maxima and minima of the function

$$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy.$$

c) Consider the function defined only on the domain

$$D = \{(x,y): 0 < y \leq ax\}, \quad 0 < a < 1$$

by $f(x,y) = \frac{1}{x-y}$.

Prove that $\lim_{|x| \to \infty} f(x,y) = 0$.

d) Prove that the function, defined by,

$$h(x,y) = \begin{cases} \frac{\sin\sqrt{1-x^2-2y^2}}{\sqrt{1-x^2-2y^2}}, & x^2 + 2y^2 < 1 \\ 1 \quad , & x^2 + 2y^2 = 1 \end{cases}$$

is continuous on the ellipse $x^2 + 2y^2 = 1$.

Unit-II

3. Answer any two of the following questions: **$2 \times 2 = 4$**

a) Evaluate $\iint xy(x+y)dxdy$ over the area between, $y = x^2$ & $y = x$.

b) State the theorem of equivalence of a double integral with repeated integral.

c) Suppose that

$$f(x,y) = \begin{cases} 0, & \text{if } x \text{ and } y \text{ are rational} \\ 1, & \text{if } x \text{ is rational and } y \text{ is irrational} \\ 2, & \text{if } x \text{ is irrational and } y \text{ is rational} \\ 3, & \text{if } x \text{ and } y \text{ are irrational} \end{cases}$$

Find $\overline{\int_R} f(x,y)d(x,y)$ and $\underline{\int_R} f(x,y)d(x,y)$ if $R = [a,b] \times [c,d]$.

4. Answer any two of the following questions: **$2 \times 4 = 8$**

a) Show that if $0 < h < 1$

$$\int_h^1 \left\{ \int_h^1 f(x,y)dx \right\} dy = \int_h^1 \left\{ \int_h^1 f(x,y)dy \right\} dx = 0 \text{ but}$$

$$\int_0^1 \left\{ \int_0^1 f(x,y)dx \right\} dy \neq \int_0^1 \left\{ \int_0^1 f(x,y)dy \right\} dx$$

where $f(x,y) = \frac{y^2-x^2}{(y^2+x^2)^2}$.

b) Show that $\iint \sqrt{x^3y^4(1-x^2-y^2)}dxdy = \frac{4\pi}{585}$ over the positive quadrant of the circle $x^2+y^2=1$.

c) Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, using Lagrange's multipliers.

Unit-III

5. **Answer any one of the following questions:** $1\times 1 = 1$

a) Prove that the vector , given by,
$\vec{F} = (6xy+z^3)\hat{\imath} + (3x^2-z)\hat{\jmath} + (3xz^2-y)\hat{k}$
represents a conservative force field.

b) Define Laplace equation.

6. **Answer any three questions of the followings:** $3\times 4 = 12$

a) Compute (interms of the constants a, b) the work done by the vector field $\vec{F} = (a\sin z + bxy^2)\hat{\imath} + 2x^2y\hat{\jmath} + (x\cos z - z^2)\hat{k}$ along the portion of helix $x=\cos t$, $y=\sin t$, $z=t$ form $(1,0,0)$ to $(1,0,2\pi)$.

b) Consider a vector field $\vec{F}(x,y) = \frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$. Suppose that c is a smooth curve in the right half space $x>0$ joining two pints $A:(x_1,y_1)$ and $B:(x_2,y_2)$. Express $\int_c \vec{F}.\overrightarrow{dr}$ interms of the polar co-ordinates (r_1,θ_1) and (r_2,θ_2) of A and B.

c) Define irrotational and solenoidal vector. If the vectors $\vec{p}$ and $\vec{q}$ be irrotational, then prove that the vector $\vec{p}\times\vec{q}$ is solenoidal.

d) Evaluate $\vec{\nabla}.(\vec{A}\times\vec{r})$ if $\vec{\nabla}\times\vec{A} = \vec{0}$, with their usual notations.

Unit-IV

7. **Answer any two from the following questions:** $2\times 2 = 4$

a) Using Stokes's theorem, prove that,

$$\text{div Curl}\vec{F} = 0 .$$

where $\vec{F}$ is any continuously differentiable vector function.

b) Show that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\oint_C (xdy - ydx).$$

c) Prove that $\oint_V \vec{\nabla}\varphi\, dV = \oint_S \varphi\,\hat{n}\, dS$, with their usual notations.

8. **Answer any two from the following questions:** $2\times 4 = 8$

a) Calculate the circulation of the vector field $\vec{F}(x,y,z) = (x^2+z)\hat{\imath} + (y^2+2x)\hat{\jmath} + (z^2-y)\hat{k}$ along the curve of the intersection of the sphere $x^2+y^2+z^2=1$ with the cone $z=\sqrt{x^2+y^2}$ traversed in the counter-clockwise direction around the z-axis when viewed from above.

b) Applying Stokes's theorem, evaluate

$\int_\Gamma \{(x+y)dx + (2x-z)dy + (y+z)dz\}$

where Γ is the boundary of the triangle with vertices (2, 0, 0) , (0, 3, 0) and (0, 0, 6).

c) State Gauss's divergence theorem. Verify Gauss's divergence theorem for the vector field $\vec{F} = y\hat{\imath} + x\hat{\jmath} + z^2\hat{k}$ over the cylindrical region bounded by $x^2+y^2=9$, $z=0, z=2$.

$1+3$