End Semester Examination 2019

IV

Core Course VIII

Full Marks: 50 **Time: 2 hours**

Unit-I

Answer any three questions: $3 \times 5 = 15$

1. If a bounded function f on $[a,b]$ has a set of points of discontinuities having finite number of limit points on $[a,b]$, then prove that f is R-integrable on $[a,b]$.
Let f be defined on $[0,2]$ by
$$f(x) = \begin{cases} 0, & x = \frac{n}{n+1}, \ \frac{n+1}{n}, \ n = 1,2,3 \ldots \ldots \\ 1, & elsewhere \end{cases}$$
Is f R-integrable on $[0,2]$? Find $\int_0^2 f(x)dx$. $3+1+1$

2. State and prove Darboux theorem. $1+4$

3. If a function $f: [a,b] \to \mathcal{R}$ be bounded on $[a,b]$ and c is a point of $[a,b]$. Then prove that f is integrable on $[a,b]$ iff f is integrable on $[a,c]$ and $[c,b]$. Also prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. $3+2$

4. State fundamental theorem of integral calculus. Let a function f is defined on $[-3,3]$ by
$$f(x) = \begin{cases} 2x\sin\frac{\pi}{x} - \pi\cos\frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
Show that f is integrable on $[-3,3]$ and find $\int_{-3}^3 f(x)dx$.
Prove that $L(x) = \int_1^x \frac{1}{t}dt$, $x > 0$ is continuous on $(0,\infty)$. $1+2+2$

Unit-II

5. **Answer any one question:** **2**
a) Evaluate $\int_0^\infty e^{-x}\,dx$ if it exists.
b) Prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

6. **Answer any one question:** **3**
a) Using the formula $\int_{-\infty}^\infty e^{-u^2}du = \sqrt{\pi}$ prove that $\Gamma(1/2) = \sqrt{\pi}$
b) Discuss the convergency of the integral
$\int_1^\infty \frac{1}{x^p}dx$ for > 1 .

Unit-III

7. **Answer any one question:** **$1 \times 2 = 2$**
a) Examine the uniform convergence of sequence of functions $\{f_n\}$ over $\mathcal{R}$ where
$f_n(x) = \frac{2n^2x^2}{e^{n^2x^2}} - 1, \forall x \in \mathcal{R}$.
b) State Weierstrass M test for uniform convergence of series of functions.

8. **Answer any two questions:** **$2 \times 4 = 8$**
a) If $\sum_{n=1}^\infty U_n$ be a series of real valued functions which converges uniformly to a sum function $S(x)$ over $[a,b]$ and each $U_n(x)$ is R-integrable over $[a,b]$, then prove that
i) $S(x)$ is R-integrable over $[a,b]$.
ii) $\int_a^b f(x)dx = \sum_{n=1}^\infty \int_a^b U_n(x)dx$. $2+2$

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b) Let $\sum_{n=1}^{\infty} U_n$ be a series of functions on $[0,1]$ where

$U_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, \forall x \in [0,1]$.

Find the sum function. Examine the uniform convergence of the series of functions on $[0,1]$.

c) Let D be a subset of $\mathcal{R}$ and a sequence of functions $\{f_n\}$ be uniform convergent on D to $f(x)$.

Let $x_0 \in D'$ and $\lim_{x\to x_0} f_n(x) = a_n,\ \forall n$. Then prove that,

i) $\{a_n\}$ is convergent.

ii) $\lim\limits_{x\to x_0} f(x) = \lim_{n\to\infty} a_n$. 2 + 2

Unit-IV

Answer any one question : 2

9. Given: $f(x) = \begin{cases} \pi + x, & -\pi < x < 0; \\ \pi - x, & 0 < x < \pi. \end{cases}$

Is the function f(x) an even function? If, so find the Fourier series for f(x).

10. Expand the function$f(x) = \pi x - x^2$ in half range sine series in $(0, \pi)$ up to three terms.

11. **Answer any two:** 4 × 2 = 8

a) If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos x + b_n \sin x)$ converges then show that $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

b) Let f be bounded and integrable in $[-\pi, \pi]$ and let it be possible to divide $[-\pi, \pi]$ into a finite number of open sub intervals in which f is monotone. Then show that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\xi + b_n \sin n\xi) = \begin{cases} \frac{1}{2}\{f(\xi - 0) + f(\xi + 0)\} \text{ for } -\pi < \xi < \pi \\ \frac{1}{2}\{f(\pi - 0) + f(-\pi + 0)\} \text{ for } \xi = \pm\pi \end{cases}$$

under the conditions that $f(x + 0)$ and $f(x - 0)$ exist and

$$a_n = \int_{-\pi}^{\pi} f(x) \cos nx dx\ ,\ x \geq 0;$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx\ ,\ x \geq 1;$$

c) State and prove Bessel's inequally.

Unit-V

12. **Answer any one question:** 1 × 2 = 2

a) Find the radius of convergence of the power series

$$x + \frac{2^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \ldots\ldots\ldots$$

b) Obtain the value of the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ by applying Abel's theorem.

13. **Answer any two questions:** 2 × 4 = 8

a) If the power series $\sum_{n=0}^{\infty} a_n x^n$ is neither nowhere convergent nor everywhere convergent, then prove that there exists a positive real number R such that the series

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converges absolutely for all x satisfying $|x| < R$ and diverges for all x satisfying $|x| > R$.

b) Assuming the power series expansion of

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \dots\dots\dots,$$

obtain the power series expansion of $\sin^{-1}x$ over $(-1,1)$. Deduce that

$$1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \dots\dots = \frac{\pi}{2}.$$

c) State Cauchy-Hadamard theorm. If $\sum_{n=0}^{\infty} a_n x^n$ be a series with non zero radius of convergence R, then prove that the series converges uniformly over $[-s, s]$, where $0 < s < R$.

2 + 2

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